ODD-GRACEFUL LABELINGS OF TREES OF DIAMETER 5

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Abstract. A difference vertex labeling of a graph G is an assignment f of labels to the vertices of G that induces for each edge xy the weight $|f(x)-f(y)|$. A difference vertex labeling f of a graph G of size n is odd-graceful if f is an injection from $V(G)$ to $\{0,1,...,2n-1\}$ such that the induced weights are $\{1,3,...,2n-1\}$. We show here that any forest whose components are caterpillars is odd-graceful. We also show that every tree of diameter up to five is odd-graceful.

1. Introduction

Let G be a graph of order m and size n, a *difference vertex labeling* of G is an assignment f of labels to the vertices of G that induces for each edge xy a label or *weight* given by the absolute value of the difference of its vertex labels. Graceful labelings are a well-known type of difference vertex labeling; a function f is a *graceful labeling* of a graph G of size n if f is an injection from $V(G)$ to the set $\{0,1,...,n\}$ such that, when each edge xy of G has assigned the weight $|f(x)-f(y)|$, the resulting weights are distinct; in other words, the set of weights is $\{1,2,...,n\}$. A graph that admits a graceful labeling is said to be *graceful*.

When a graceful labeling f of a graph G has the property that there exists an integer λ such that for each edge xy of G either $f(x)\le\lambda<f(y)$ or $f(y)\le\lambda<f(x)$, f is named an α-labeling and G is said to be an α-graph. From the definition it is possible to deduce that an α-graph is necessarily bipartite and that the number λ (called the *boundary value* of f) is the smaller of the two vertex labels that yield the edge with weight 1. Some examples of α-graphs are the cycle C_n when $n\equiv 0(\operatorname{mod} 4)$, the complete bipartite graph $K_{m,n}$, and caterpillars (i.e., any tree with the property that the removal of its end vertices leaves a path).

A little less restrictive than α-labelings are the odd-graceful labelings introduced by Gnanajothi in 1991 [4]. A graph G of size n is *odd-graceful* if there is an injection $f:V(G)\to\{0,1,2,...,2n-1\}$ such that the set of induced weights is $\{1,3,...,2n-1\}$. In this case, f is said to be an *odd-graceful labeling* of G. One of the applications of these labelings is that trees of size n, with a suitable odd-graceful labeling, can be used to generate cyclic decompositions of the complete bipartite graph $K_{n,n}$. In Figure 1 we show an odd-graceful tree of size 6 together with its embedding in the circular arrangement used to produce the cyclic decomposition of $K_{6,6}$. Once the labeled tree has been embedded, succesives 30° (counterclockwise) rotations produce the desired cyclic decomposition of $K_{6,6}$.

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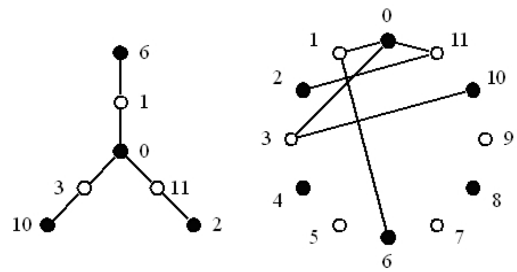

Fig. 1. Cyclic decomposition of $K_6, 6$

Gnanajothi [4] proved that the class of odd-graceful graphs lies between the class of α-graphs and the class of bipartite graphs; she proved that every α-graph is also odd-graceful. The reverse case does not work, for example the odd-graceful tree shown in Figure 1 is the smallest tree without an α-labeling. Since many families of α-graphs are known, the most attractive examples of odd-graceful graphs are those without an α-labeling or where an α-labeling is unknown; for instance, Gnanajothi [4] proved that the following are odd-graceful graphs: C_n when $n \equiv 2 (\mathrm{mod}\, 4)$, the disjoint union of C_4, the prism $C_n \times K_2$ if and only if n is even, and trees of diameter 4 among others. Eldergil [2] proved that the one-point union of any number of copies of C_6 is odd-graceful. Seoud, Diab, and Elsakhawi [5] showed that a connected n-partite graph is odd-graceful if and only if $n = 2$ and that the join of any two connected graphs is not odd-graceful.

A detailed account of results in the subject of graph labelings can be found in Gallian' survey [3].

Gnanajothi [4] conjectured that all trees are odd-graceful and verified this conjecture for all trees with order up to 10. The author has extended this up to trees with order up to 12[1]. In this paper we prove that all trees of diameter 5 are odd-graceful and that any forest whose components are caterpillars is odd-graceful.

2. Odd-Graceful Forests

In this section we study *forests* that accept odd-graceful labelings. Recall that a forest with more than one component cannot be graceful bucause it has "too many edges". First we prove that any graph that admits an α-labeling also admits an odd-graceful labeling by transforming conviniently its α-labeling.

Theorem 1. *Any α-graph is odd-graceful.*

Proof. Let G be an α-graph of size n, as consequence G is bipartite with partition $\{A, B\}$. Suppose that f is an α-labeling of G such that $\max\{f(x) : x \in A\} < \min\{f(x) : x \in B\}$. Let g be a labeling of the vertices of G defined by

$$g(x) = \begin{cases} 2f(x), & x \in A \\ 2f(x) - 1, & x \in B. \end{cases}$$

Thus, the labels assigned by g are in the set $\{0, 1, ..., 2n-1\}$, furthermore, the weight of the edge xy of G induced by the labeling f, where $x \in A$ and $y \in B$, is $w = f(y) - f(x)$, so its weight under the labeling g is $g(y) - g(x) = 2f(y) - 1 -$

[1]Odd-graceful labelings of trees of order 11 and 12 can be found at http://cims.clayton.edu/cbarrien/research

$2f(x) = 2(f(y) - f(x)) - 1 = 2w - 1$. Since $1 \leq w \leq n$, we have that the weights induced by g are $\{1, 3, ..., 2n-1\}$. Therefore, g is an odd-graceful labeling of G. $\square$

In Figure 2 we show an example of an α-labeling of a caterpillar, followed for the corresponding odd-graceful labeling. We use this labeling in the next theorem.

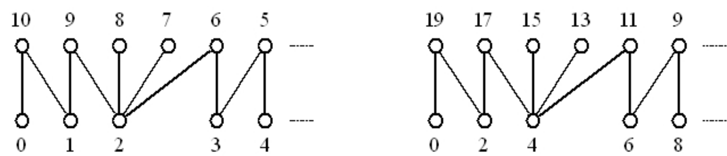

Fig. 2. Odd-graceful labeling of a caterpillar

Theorem 2. *Any forest which components are caterpillars is odd-graceful.*

Proof. Let F_i be a caterpillar of size $n_i \geq 1$, for $1 \leq i \leq k$. Let $u_i, v_i \in V(F_i)$ such that $d(u_i, v_i) = diam(F_i)$; so identifying v_i with u_{i+1}, for each $1 \leq i \leq k-1$, we have a caterpillar F of size $\sum_{i=1}^{k} n_i = n$. Now we proceed to find both, the α-labeling of F and its corresponding odd-graceful labeling, using the scheme shown in Figure 2. Once the odd-graceful labeling has been obtained, we disengage each caterpillar F_i from F, keeping their labels; in this form, the weights induced are $\{1, 3, ..., 2n-1\}$. To eliminate the overlapping of labels we subtract 1 from each vertex label of F_i when i is even, in this way the weights remain the same and the labels assigned on $u_{i+1} and\ v_i$ differ by one unit. Therefore, the labeling of the forest $\bigcup_{i=1}^{k} F_i$ is odd-graceful. $\square$

In Figure 3 we show an example of this construction using the odd-graceful labeling obtained in Figure 2.

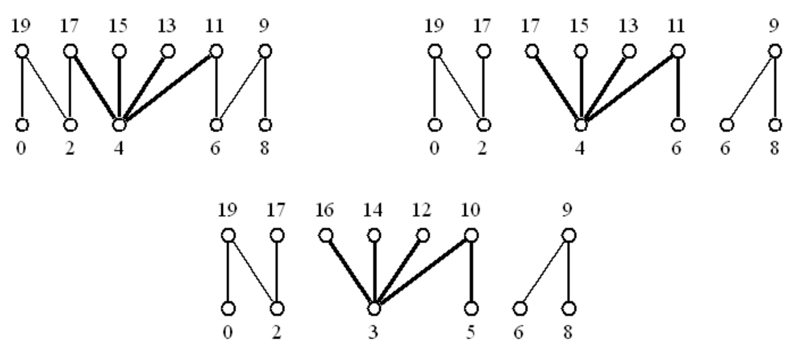

Fig. 3. Odd-graceful labeling of a caterpillar

The procedure used in this proof can be extended to the disjoint union of graphs with α-labelings. In fact, suppose that the concatenation of blocks $B_1, B_2, ..., B_k$ results in a graph G whose block-cutpoint graph is a path. In [1] we proved that if each B_i is an α-graph, so it is G. Transforming this α-labeling into an odd-graceful labeling and disconnecting G into blocks, the disjoint union of these blocks is odd-graceful.

Theorem 3. *The disjoint union of blocks that accept α-labelings is odd-graceful.*

3. Odd-Graceful Trees of Diameter Five

Every tree of diameter at most 3 is a caterpillar, therefore it is odd-graceful. Gnanajothi [4] proved that every rooted tree of height 2 (that is, diameter 4) is odd-graceful. In the next theorem we represent trees of diameter 5 as rooted trees of height 3 and prove that they are odd-graceful.

Let T be a tree of diameter 5; T can be represented as a rooted tree of height 3 by using any of its two central vertices as the root vertex. Note that only one of the vertices in level 1 has descendants in level 3; this vertex will be located in the right extreme of level 1. Now, within each level, the vertices are placed from left to right in such a way that their degrees are increasing. In the proof of the next theorem we use this type of representation of T, that is, assuming that v (one of the two central vertices) is the root.

Theorem 4. *All trees of diameter five are odd-graceful.*

Proof. Let T be a tree of diameter 5 and size n. Suppose that T has been drawn according to the previous description. Let $v_{i,j}$ denote the ith vertex of level j, for $j = 1, 2, 3$, this vertex is placed at the right of $v_{i+1,j}$. Consider the labeling f of the vertices within each level given by recurrence as follows: $f(v) = 0$, $f(v_{1,1}) = 2n - 2\deg(v) + 1$, $f(v_{1,2}) = 2$, $f(v_{1,3}) = 3$, and $f(v_{i,j}) = f(v_{i-1,j}) + d(v_{i,j}, v_{i-1,j})$ where $i \geq 2$ and $1 \leq j \leq 3$.

We claim that f is an odd-graceful labeling of T. In fact, let us see that there is no overlapping of labels. On level 0 the label used is 0 and on level 2 all labels are even being 2 the smallest label used here. On levels 1 and 3 the labels used are odd; on level 1 the labels used are $2n - 1, 2n - 3, ..., 2n - 2\deg(v) + 1$, while on level 3 the labels used are $3, 5, ..., 2\deg(v_{1,2}) - 1$. Now we need to prove that $2n - 2\deg(v) + 1 > 2\deg(v_{1,2}) - 1$; since T is a tree of diameter 5, at least two vertices on level 1 has descendants, so $n + 1 > \deg(v) + \deg(v_{1,2})$, which implies the desired inequality.

As a consequence of the fact that labels used in consecutive levels have different parity, each weight obtained is an odd number not exceeding $2n - 1$. Suppose that $v_{i+1,j}$ and $v_{i,j}$ have the same father x, by definition of f, the edges $xv_{i+1,j}$ and $xv_{i,j}$ have consecutive weights. If $v_{i+1,j}$ and $v_{i,j}$ have different father, x and y respectively, then $|f(y) - f(v_{i,j})| = |(f(x) + 2) - (f(v_{i+1,j}) + 4)| = |f(x) - f(v_{i+1,j}) - 2|$. Thus, the weights are $2n - 2\deg(v) - 1, ..., 2\deg(v) + 1$. On level 2, the weights are $2n - 2\deg(v) - 3, ..., 2\deg(v_{1,2}) - 1$, and on level 3 the weights are $2\deg(v_{1,2}) - 3, ..., 1$.

Therefore, f is an odd-graceful labeling of T. □

In Figure 4 we present a scheme of this labeling for a tree of size 13.

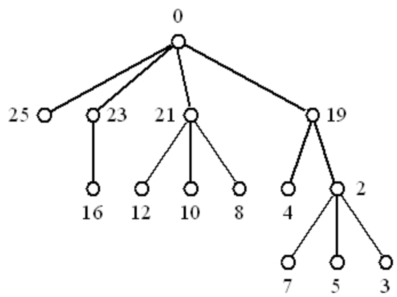

Fig. 4. Odd-graceful tree of diameter 5

Similar arguments can be used to find odd-graceful labelings of trees of diameter 6; however we do not have a general labeling scheme for this case. So it is an open problem determining whether trees of diameter 6 are odd-graceful. In Figure 5, we give an example of an odd-graceful labeling for a tree of size 17 and diameter 6.

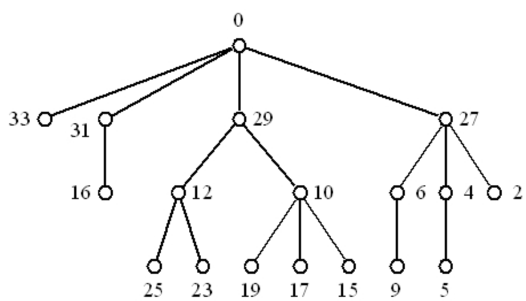

Fig. 5. Odd-graceful tree of diameter 6

To conclude this section, we show in Figure 6 an odd-graceful labeling for a special type of tree of diameter 6, namely the star $S(n,3)$ with n spokes of length 3.

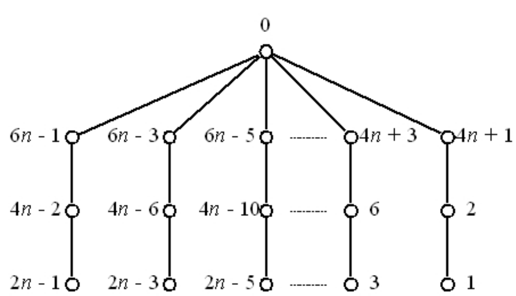

Fig. 6. Odd-graceful labeling of the star $S(n,3)$

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